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New congruences for overcubic partition pairs

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Abstract

In this paper, we study congruence properties of overcubic partition pairs. Let $\overline{b}(n)$ denote the number of overcubic partition pairs of n. We will establish some new Ramanujan type congruences and several infinite families of congruences modulo powers of 2 satisfied by $\overline{b}(n)$.

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1 Introduction

H. C. Chan in his papers [3, 4, 5] studied the congruence properties of the cubic partition function $a(n)$, which is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{1}{f_1 f_2}. \tag{1.1}$$

Here and throughout this paper, for any positive k, f_k is defined by

$$f_k := \prod_{i=1}^{\infty}(1-q^{ki}) = (q^k;q^k)_\infty. \tag{1.2}$$

Following H. C. Chan, B. Kim [7] studied its overpartiton analog in which the overcubic partition function $\overline{a}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{a}(n)q^n = \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{f_4}{f_1^2 f_2}. \tag{1.3}$$

Hirschhorn [6] gave an elementary proof of the results satisfied by $\overline{a}(n)$, which appeared in Kim's paper [7], and Sellers [11] proved several arithmetic properties satisfied by $\overline{a}(n)$ by employing elementary generating function methods. Zhao and Zhong [13] established congruences modulo $5, 7$ and 9 for the partition function $b(n)$, defined by

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty^2} = \frac{1}{f_1^2 f_2^2}. \tag{1.4}$$

Kim [8] introduced two partition statistics to explain the congruences modulo 5 and 7 for $b(n)$. Since $b(n)$ counts a pair of cubic partitions, Kim [8] called $b(n)$, the number of cubic partition

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pairs. Recently, Kim [9] studied congruence properties of $\overline{b}(n)$ whose generating function is given by

$$\sum_{n=0}^{\infty}\overline{b}(n)q^n = \frac{(-q;q)_\infty^2(-q^2;q^2)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty^2} = \frac{f_4^2}{f_1^4 f_2^2}, \tag{1.5}$$

where $\overline{b}(n)$ denote the number of overcubic partition pairs of n. Using arithmetic properties of quadratic forms and modular forms, Kim [9] derived the following two congruences

$$\overline{b}(8n+7) \equiv 0 \pmod{2^6}, \tag{1.6}$$

$$\overline{b}(9n+3) \equiv 0 \pmod{3}. \tag{1.7}$$

The congruence (1.6) appears to be incorrect. It seems that it is true for modulo 2^5.

More recently, Lin [10] studied various arithmetic properties of $\overline{b}(n)$ modulo 3 and 5. For example, he proved for any $\alpha \geq 2$, $n \geq 0$

$$\overline{b}(3^\alpha(3n+2)) \equiv 0 \pmod{3} \tag{1.8}$$

and for $\alpha \geq 0$,

$$\overline{b}(380 \cdot 5^\alpha) \equiv 0 \pmod{5}. \tag{1.9}$$

With this motivation, we establish some new Ramanujan like congruences and infinite families of congruences modulo powers of 2 for overcubic partition pairs $\overline{b}(n)$ by using some elementary generating function and dissection formulas. Our main results can be stated as follows

Theorem 1.1. For all $n \geq 0$, we have

$$\overline{b}(4n+3) \equiv 0 \pmod{2^4}, \tag{1.10}$$

$$\overline{b}(8n+7) \equiv 0 \pmod{2^5}, \tag{1.11}$$

$$\overline{b}(32n+12) \equiv 0 \pmod{2^5}, \tag{1.12}$$

$$\overline{b}(32n+24) \equiv 0 \pmod{2^5}, \tag{1.13}$$

$$\overline{b}(8n+6) \equiv 0 \pmod{2^6}, \tag{1.14}$$

$$\overline{b}(32n+28) \equiv 0 \pmod{2^6}, \tag{1.15}$$

$$\overline{b}(64n+56) \equiv 0 \pmod{2^6}, \tag{1.16}$$

$$\overline{b}(16n+14) \equiv 0 \pmod{2^7}. \tag{1.17}$$

Theorem 1.2. For $\alpha \geq 0$, $n \geq 0$ and $r_1 \in \{14, 46, 62, 78\}$, we have

$$\overline{b}(16 \cdot 5^{2\alpha+2}n + r_1 \cdot 5^{2\alpha+1}) \equiv 0 \pmod{2^7}. \tag{1.18}$$

Theorem 1.3. For $\alpha \geq 0$, $n \geq 0$ and $r_2 \in \{10, 26, 58, 74, 90, 106\}$, we have

$$\overline{b}(16 \cdot 7^{2\alpha+2}n + r_2 \cdot 7^{2\alpha+1}) \equiv 0 \pmod{2^7}. \tag{1.19}$$

Theorem 1.4. For any $\alpha \geq 0$, $n \geq 0$, we have

$$\overline{b}(16 \cdot 3^{4\alpha+4}n + 10 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8}, \tag{1.20}$$

$$\overline{b}(16 \cdot 3^{4\alpha+4}n + 26 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8}. \tag{1.21}$$

Theorem 1.5. For any $\alpha \geq 0$, $n \geq 0$, we have

$$\bar{b}(8 \cdot 3^{4\alpha+4}n + 5 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^6}, \tag{1.22}$$
$$\bar{b}(8 \cdot 3^{4\alpha+4}n + 13 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^6}. \tag{1.23}$$

2 Preliminary results

For $|ab| < 1$, we define Ramanujan's general theta function $f(a,b)$, as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.1}$$

The special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{2.3}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1, \tag{2.4}$$

where the product representations arise from Jacobi's triple product identity [2, p.35, Entry 19]

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{2.5}$$

We need the following dissection formulas to prove our main results,

Lemma 2.1. The following 2-dissections hold

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.6}$$
$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.7}$$

Lemma 2.1. is a consequence of dissection formulas of Ramanujan, collected in Berndt's book [2, p. 40, Entry 25].

3 Proof of Theorem 1.1.

To prove Theorem 1.1. we first establish the following lemma.

Lemma 3.1. We have

$$\sum_{n=0}^{\infty} \overline{b}(4n)q^n = \frac{f_2^{52}}{f_1^{40}f_4^{16}} + 96q\frac{f_2^{28}}{f_1^{32}} + 256q^2\frac{f_2^4 f_4^{16}}{f_1^{24}}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \overline{b}(4n+2)q^n = 16\frac{f_2^{40}}{f_1^{36}f_4^{8}} + 256q\frac{f_2^{16} f_4^{8}}{f_1^{28}}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \overline{b}(4n+1)q^n = 4\frac{f_2^{46}}{f_1^{38}f_4^{12}} + 192q\frac{f_2^{22} f_4^{4}}{f_1^{30}}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} \overline{b}(4n+3)q^n = 48\frac{f_2^{34}}{f_1^{34}f_4^{4}} + 256q\frac{f_2^{10} f_4^{12}}{f_1^{26}}. \tag{3.4}$$

Proof. Substituting (2.7) into (1.5),

$$\sum_{n=0}^{\infty} \overline{b}(n)q^n = \frac{f_4^2}{f_2^2}\frac{1}{f_1^4} = \frac{f_4^{16}}{f_2^{16}f_8^4} + 4q\frac{f_4^4 f_8^4}{f_2^{12}}, \tag{3.5}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{b}(2n)q^n = \frac{f_2^{16}}{f_1^{16}f_4^4} \tag{3.6}$$

and

$$\sum_{n=0}^{\infty} \overline{b}(2n+1)q^n = 4\frac{f_2^4 f_4^4}{f_1^{12}}. \tag{3.7}$$

From (2.7), we have

$$\frac{1}{f_1^{16}} = \frac{f_4^{56}}{f_2^{56}f_8^{16}} + 16q\frac{f_4^{44}}{f_2^{52}f_8^{8}} + 96q^2\frac{f_4^{32}}{f_2^{48}} + 256q^3\frac{f_4^{20}f_8^{8}}{f_2^{44}} + 256q^4\frac{f_4^{8}f_8^{16}}{f_2^{40}} \tag{3.8}$$

and

$$\frac{1}{f_1^{12}} = \frac{f_4^{42}}{f_2^{42}f_8^{12}} + 12q\frac{f_4^{30}}{f_2^{38}f_8^{4}} + 48q^2\frac{f_4^{18}f_8^{4}}{f_2^{34}} + 64q^3\frac{f_4^{6}f_8^{12}}{f_2^{30}}. \tag{3.9}$$

Substituting (3.8) and (3.9) respectively into (3.6) and (3.7), we find that

$$\sum_{n=0}^{\infty} \overline{b}(2n)q^n = \frac{f_4^{52}}{f_2^{40}f_8^{16}} + 16q\frac{f_4^{40}}{f_2^{36}f_8^{8}} + 96q^2\frac{f_4^{28}}{f_2^{32}} + 256q^3\frac{f_4^{16}f_8^{8}}{f_2^{28}} + 256q^4\frac{f_4^{4}f_8^{16}}{f_2^{24}} \tag{3.10}$$

and

$$\sum_{n=0}^{\infty} \overline{b}(2n+1)q^n = 4\frac{f_4^{46}}{f_2^{38}f_8^{12}} + 48q\frac{f_4^{34}}{f_2^{34}f_8^{4}} + 192q^2\frac{f_4^{22}f_8^{4}}{f_2^{30}} + 256q^3\frac{f_4^{10}f_8^{12}}{f_2^{26}}. \tag{3.11}$$

Lemma 3.1. follows from (3.10) and (3.11). This completes the proof. Q.E.D.

Proof of Theorem 1.1. By binomial theorem for any positive integers k and m, it is easy to see that

$$f_{2m}^{2^{k-1}} \equiv f_m^{2^k} \pmod{2^k}. \tag{3.12}$$

In view of (3.12), (3.2) can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{b}(4n+2)q^n &\equiv 16\frac{f_2^{24}}{f_1^4 f_4^8} + 256q f_2^2 f_4^8 \\ &\equiv 16\frac{f_2^{24}}{f_4^8}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right) + 256q f_2^2 f_4^8 \pmod{2^9},\end{aligned} \tag{3.13}$$

which implies that

$$\sum_{n=0}^{\infty} \bar{b}(8n+2)q^n \equiv 16\frac{f_1^{10} f_2^6}{f_4^4} \pmod{2^9} \tag{3.14}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{b}(8n+6)q^n &\equiv 64\frac{f_1^{14} f_4^4}{f_2^6} + 256 f_1^2 f_2^8 \\ &\equiv 64 f_2^2 f_4^4\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) + 256 f_2^9 \pmod{2^9}\end{aligned} \tag{3.15}$$

Congruence (1.14) follows from (3.15). Equating odd and even powers of q from both sides of (3.15), we obtain

$$\sum_{n=0}^{\infty} \bar{b}(16n+6)q^n \equiv 64\frac{f_2^4 f_4^5}{f_1^3 f_8^2} + 256 f_1^9 \pmod{2^9} \tag{3.16}$$

and

$$\sum_{n=0}^{\infty} \bar{b}(16n+14)q^n \equiv 128\frac{f_2^6 f_8^2}{f_1^3 f_4} \pmod{2^9}. \tag{3.17}$$

Congruence (1.17) is immediate from (3.17).

From (3.4), we have

$$\sum_{n=0}^{\infty} \bar{b}(4n+3)q^n \equiv 48\frac{f_2^{34}}{f_1^{34} f_4^4} \pmod{2^8}. \tag{3.18}$$

Since $f_{2m}^8 \equiv f_m^{16} \pmod{2^4}$, (3.18) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{b}(4n+3)q^n &\equiv 48\frac{f_2^{18}}{f_1^2 f_4^4} \\ &\equiv 48\frac{f_2^{13} f_8^5}{f_4^4 f_{16}^2} + 96q\frac{f_2^{13} f_{16}^2}{f_4^2 f_8} \pmod{2^8}.\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} \bar{b}(8n+3)q^n \equiv 48\frac{f_1^{13} f_4^5}{f_2^4 f_8^2} \pmod{2^8} \tag{3.19}$$

and

$$\sum_{n=0}^{\infty}\overline{b}(8n+7)q^n \equiv 96\frac{f_1^{13}f_8^2}{f_2^2f_4} \pmod{2^8}. \tag{3.20}$$

Congruences (1.10) and (1.11) follow from (3.18) and (3.20).

In the view of (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(4n)q^n &\equiv \frac{f_2^{52}}{f_1^{40}f_4^{16}} + 96q\frac{f_2^{28}}{f_1^{32}} \\ &\equiv \frac{f_2^{52}}{f_4^{16}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^{10} + 96f_2^{12} \pmod{2^8}.\end{aligned}$$

After simplification, we find that

$$\sum_{n=0}^{\infty}\overline{b}(8n)q^n \equiv \frac{f_2^{124}}{f_1^{88}f_4^{40}} + 208q\frac{f_2^{100}}{f_1^{80}f_4^{24}} \pmod{2^8} \tag{3.21}$$

and

$$\sum_{n=0}^{\infty}\overline{b}(8n+4)q^n \equiv 40\frac{f_2^{112}}{f_1^{84}f_4^{32}} + 96f_1^{12} \pmod{2^8}. \tag{3.22}$$

Substituting (2.7) into (3.21), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(8n)q^n &\equiv \frac{f_2^{124}}{f_4^{40}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^{22} + 208q\frac{f_2^{100}}{f_4^{24}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^{20} \\ &\equiv \frac{f_4^{268}}{f_2^{184}f_8^{88}} + 40q\frac{f_4^{256}}{f_2^{180}f_8^{80}} + 112q^2\frac{f_4^{244}}{f_2^{176}f_8^{72}} \pmod{2^8},\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty}\overline{b}(16n)q^n \equiv \frac{f_2^{268}}{f_1^{184}f_4^{88}} + 112q\frac{f_2^{244}}{f_1^{176}f_4^{72}} \pmod{2^8} \tag{3.23}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(16n+8)q^n &\equiv 40\frac{f_2^{256}}{f_1^{180}f_4^{80}} \\ &\equiv 40f_2^{16}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^5 \\ &\equiv 40\frac{f_4^{70}}{f_2^{54}f_8^{20}} + 32q\frac{f_4^{58}}{f_2^{50}f_8^{12}} \pmod{2^8}.\end{aligned}$$

Extracting the terms containing q^{2n+1} from both sides of the above equation, we obtain

$$\sum_{n=0}^{\infty}\overline{b}(32n+24)q^n \equiv 32\frac{f_2^{58}}{f_1^{50}f_4^{12}} \equiv 32\frac{f_2^{10}}{f_1^2} \pmod{2^8}. \tag{3.24}$$

Congruence (1.13) follows from (3.24). Again, substituting (2.6) into (3.24) and extracting the odd and even parts of the resulting equation, we find that

$$\sum_{n=0}^{\infty} \bar{b}(64n+24)q^n \equiv 32\frac{f_1^5 f_4^5}{f_8^2} \pmod{2^8} \tag{3.25}$$

and

$$\sum_{n=0}^{\infty} \bar{b}(64n+56)q^n \equiv 64\frac{f_1^5 f_2^2 f_8^2}{f_4} \pmod{2^8}. \tag{3.26}$$

Congruence (1.16) follows from (3.26).

Substituting (2.7) into (3.22), we find that

$$\sum_{n=0}^{\infty} \bar{b}(8n+4)q^n \equiv 40\frac{f_4^{262}}{f_2^{182} f_8^{84}} + 96\frac{f_4^{42}}{f_2^{42} f_8^{12}} + 128q\frac{f_4^{30}}{f_2^{38} f_8^{4}} + 32q\frac{f_4^{250}}{f_2^{178} f_8^{76}} \pmod{2^8},$$

which implies that

$$\sum_{n=0}^{\infty} \bar{b}(16n+4)q^n \equiv 40\frac{f_2^{262}}{f_1^{182} f_4^{84}} + 96\frac{f_2^{42}}{f_1^{42} f_4^{12}} \pmod{2^8} \tag{3.27}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{b}(16n+12)q^n &\equiv 128\frac{f_2^{30}}{f_1^{38} f_4^{4}} + 32\frac{f_2^{250}}{f_1^{178} f_4^{76}}\\ &\equiv 128 f_2^3 + 32\frac{f_2^{10}}{f_1^2}\\ &\equiv 128 f_2^3 + 32\frac{f_2^5 f_8^5}{f_{16}^2} + 64q\frac{f_2^5 f_4^2 f_{16}^2}{f_8} \pmod{2^8},\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} \bar{b}(32n+12)q^n \equiv 128 f_1^3 + 32\frac{f_1^5 f_4^5}{f_8^2} \pmod{2^8} \tag{3.28}$$

and

$$\sum_{n=0}^{\infty} \bar{b}(32n+28)q^n \equiv 64\frac{f_1^5 f_2^2 f_8^2}{f_4} \pmod{2^8}. \tag{3.29}$$

Congruences (1.12) and (1.15) follow from (3.28) and (3.29). Q.E.D.

4 Proof of Theorem 1.2.

In view of (3.12) and (3.16), we have

$$\sum_{n=0}^{\infty} \bar{b}(16n+6)q^n \equiv 64\frac{f_2^4 f_4^5}{f_1^3 f_8^2} \equiv 64\frac{f_2^6}{f_1^3} \pmod{2^7}. \tag{4.1}$$

By the definition of $\psi(q)$, (4.1) becomes

$$\sum_{n=0}^{\infty} \overline{b}(16n+6)q^n \equiv 64\psi^3(q) \pmod{2^7}. \tag{4.2}$$

From [2, p.49, Corollary], we have

$$\psi(q) = f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}). \tag{4.3}$$

Employing (4.3) into (4.2), we obtain

$$\sum_{n=0}^{\infty} \overline{b}(16n+6)q^n \equiv 64(f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}))^3 \pmod{2^7}.$$

After simplification, extracting the terms containing q^{5n+4} from both sides of the resulting equation, we get

$$\sum_{n=0}^{\infty} \overline{b}(16(5n+4)+6)q^n \equiv 64q\psi(q^5)^3 \pmod{2^7},$$

which implies that

$$\sum_{n=0}^{\infty} \overline{b}(16(5^2n+9)+6)q^n \equiv 64\psi(q)^3 \pmod{2^7} \tag{4.4}$$

and

$$\sum_{n=0}^{\infty} \overline{b}(16(5^2n+5i+4)+6)q^n \equiv 0 \pmod{2^7}, \quad i = 0, 2, 3, 4. \tag{4.5}$$

Combining (4.2) and (4.4), we see that

$$\overline{b}(16(5^2n+9)+6) \equiv \overline{b}(16n+6) \pmod{2^7}. \tag{4.6}$$

Therefore, by induction on α, we have

$$\overline{b}(16 \cdot 5^{2\alpha}n + 6 \cdot 5^{2\alpha}) \equiv \overline{b}(16n+6) \pmod{2^7}. \tag{4.7}$$

Congruence (1.18) follows from (4.5) and (4.7).

5 Proof of Theorem 1.3.

From Entry 17(iv) on page 303 in Berndt's book [2], we have the 7-dissection:

$$\psi(q) = f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3f(q^7, q^{42}) + q^6\psi(q^{49}). \tag{5.1}$$

Substituting (5.1) into (4.2), we have

$$\sum_{n=0}^{\infty} \overline{b}(16n+6)q^n \equiv 64(f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3f(q^7, q^{42}) + q^6\psi(q^{49}))^3 \pmod{2^7}.$$

Simplifying and extracting the terms containing q^{7n+4} from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \bar{b}(16(7n+4)+6)q^n \equiv 64q^2\psi(q^7)^3 \pmod{2^7},$$

which implies that

$$\sum_{n=0}^{\infty} \bar{b}(16(7^2n+18)+6)q^n \equiv 64\psi(q)^3 \pmod{2^7} \tag{5.2}$$

and

$$\sum_{n=0}^{\infty} \bar{b}(16(7^2n+7i+4)+6)q^n \equiv 0 \pmod{2^7}, \quad i=0,1,3,4,5,6. \tag{5.3}$$

Combining (4.2) and (5.2), we see that

$$\bar{b}(16(7^2n+18)+6) \equiv \bar{b}(16n+6) \pmod{2^7}. \tag{5.4}$$

So, by induction on α, we have

$$\bar{b}(16\cdot 7^{2\alpha}n+6\cdot 7^{2\alpha}) \equiv \bar{b}(16n+6) \pmod{2^7}. \tag{5.5}$$

Congruence (1.19) follows form (5.3) and (5.5).

6 Proof of Theorem 1.4.

In view of (3.12) and (3.17),we have

$$\sum_{n=0}^{\infty} \bar{b}(16n+14)q^n \equiv 128\psi^7(q) \pmod{2^8}. \tag{6.1}$$

Recently, X. Yang et al. [12] have proved the following results:

for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} f\left(3^{4\alpha}n+\frac{7\cdot 3^{4\alpha}-7}{8}\right)q^n \equiv \psi^7(q) \pmod{4}$$

and

$$\sum_{n=0}^{\infty} f\left(3^{4\alpha+3}n+\frac{5\cdot 3^{4\alpha+3}-7}{8}\right)q^n \equiv q^2\psi^7(q^3) \pmod{4}. \tag{6.2}$$

Therefore, from (6.1) and (6.2), we see that

$$\sum_{n=0}^{\infty} \bar{b}(16\cdot 3^{4\alpha+3}n+10\cdot 3^{4\alpha+3})q^n \equiv 128q^2\psi^7(q^3) \pmod{2^8}. \tag{6.3}$$

Equating the terms involving q^{3n} and q^{3n+1} from both sides of (6.3), we obtain

$$\bar{b}(16\cdot 3^{4\alpha+3}(3n)+10\cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8},$$
$$\bar{b}(16\cdot 3^{4\alpha+3}(3n+1)+10\cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8}.$$

This completes the proof.

7 Proof of Theorem 1.5.

Combining (3.12) and (3.20), we obtain

$$\sum_{n=0}^{\infty} \overline{b}(8n+7)q^n \equiv 32\psi^7(q) \pmod{2^6}. \tag{7.1}$$

Remaining part of the proof is similar to the proof of Theorem 1.4., hence we omit the details.

8 Internal congruences for $\overline{b}(n)$.

Theorem 8.1. For $n \geq 0$, we have

$$\overline{b}(64n+24) \equiv \overline{b}(32n+12) \pmod{2^6},$$
$$\overline{b}(64n+56) \equiv \overline{b}(32n+28) \pmod{2^8}.$$

Proof. First congruence relation follows from (3.25) and (3.28) and second one follows from (3.26) and (3.29). Q.E.D.

Theorem 8.2. For $n \geq 0$, we have

$$\overline{b}(100n+10) \equiv 96\overline{b}(20n+2) \pmod{2^8},$$
$$\overline{b}(100n+30) \equiv 96\overline{b}(20n+6) \pmod{2^8},$$
$$\overline{b}(100n+70) \equiv 96\overline{b}(20n+14) \pmod{2^8},$$
$$\overline{b}(100n+90) \equiv 96\overline{b}(20n+18) \pmod{2^8}.$$

Proof. From (3.2), we have

$$\sum_{n=0}^{\infty} \overline{b}(4n+2)q^n = 16\frac{f_2^{40}}{f_1^{36}f_4^8} \equiv 16\psi^4(q) \pmod{2^8}. \tag{8.1}$$

Substituting (4.3) into (8.1) and then extracting the terms involving q^{5n+2} from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{b}(20n+10)q^n \equiv 16(f(q^2,q^3)^2 f(q,q^4)^2 + 12qf(q^2,q^3)f(q,q^4)\psi^2(q^5) + q^2\psi^4(q^5)) \pmod{2^8}. \tag{8.2}$$

From [1, p.26, (1.6.7)], we have

$$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^3). \tag{8.3}$$

Applying (8.3) into (8.2), we see that

$$\sum_{n=0}^{\infty} \overline{b}(20n+10)q^n \equiv -80q^2\psi^4(q^5) + 96\psi^4(q) \pmod{2^8},$$

which implies that

$$\sum_{n=0}^{\infty}\overline{b}(20n+10)q^n - 96\sum_{n=0}^{\infty}\overline{b}(4n+2)q^n \equiv -80q^2\psi^4(q^5) \pmod{2^8}. \tag{8.4}$$

Extracting the terms involving q^{5n+2} from both sides of the above equation, we obtain

$$\overline{b}(100n+50) - 96\overline{b}(20n+10) \equiv -80\overline{b}(4n+2) \pmod{2^8}.$$

Theorem follows by equating the powers containing q^{5n+i} for $i = 0, 1, 3, 4$ form both sides of (8.4). Q.E.D.

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References

[1] G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebooks*, Part I. Springer-Verlag, New York, 2005.

[2] B.C. Berndt, *Ramanujan's Notebooks*, Part III. Springer-Verlag, New York, 1991.

[3] H.C. Chan, *Ramanujan's cubic continued fraction and an analog of his most beautiful identity.* Int. J. Number Theory 6 no. 3(2010), 673-680.

[4] H.C. Chan, *Ramanujan's cubic continued fraction and Ramanujan type congruences for a certain partition function.* Int. J. Number Theory, 6 no. 4(2010), 819-834.

[5] H.C. Chan, *Distribution of a certain partition function modulo powers of primes.* Acta Math. Sin., 27(2011), 625-634.

[6] M.D. Hirschhorn, *A note on overcubic partitions.* New Zealand J. Math., 42(2012), 229-234.

[7] B. Kim, *The overcubic partition function mod* 3. Ramanujan Rediscovered, Ramanujan Math. Soc. Lect. Notes Ser., 14(2010), 157-163.

[8] B. Kim, *Partition statistics for cubic partition pairs.* Electron. J. Combin., 18(2011), # P128.

[9] B. Kim, *On partition congruences for overcubic partition pairs.* Commun. Korean Math. Soc., 27(2012), 477-482.

[10] B. L. S. Lin, *Arithmetic properties of overcubic partition pairs.* Electron. J. Combin. 21(3)(2014), #P3.35.

[11] J. A. Sellers, *Elementary proofs of congruences for the cubic and overcubic partition functions.* Australas. J. Combin. 60(2)(2014), 191-197.

[12] X. Yang, S. P. Cui and B. L. S. Lin, *Overpartition function modulo powers of 2*. Ramanujan J. DOI: 10.1007/s11139-016-9784-2.

[13] H. Zhao and Z. Zhong, *Ramanujan type congruences for a partition function*. Electron. J. Combin. 18(2011), #P58.